Math 120A: Homework 3

Due: October 24, 2014

1. Read sections 2.2-3, 3.1 in Pressley.

2. Do problems 2.1.1, 2.1.2, 2.1.4, 2.2.3, 2.2.5, 2.2.8, and 2.2.9 in Pressley.

   [Note 1: It is definitely possible to do too much work for 2.1.1. What is the first thing you should check when beginning your computations?]

   [Note 2: It may be helpful to do the question below before 2.2.3.]

3. An isometry $M : \mathbb{R}^n \to \mathbb{R}^n$ is a map that preserves distance, i.e. $||M(\mathbf{v}) - M(\mathbf{w})|| = ||\mathbf{v} - \mathbf{w}||$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

   - Let M be an isometry with $M(\mathbf{0}) = \mathbf{0}$. Let $\mathbf{e}_i$ be the ith standard unit vector, so $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the ith position. Show that if $M(e_i) = e_i$ for all i, then M is the identity. [Hint: Expand the equality $||\mathbf{v} - \mathbf{e}_i||^2 = ||F(\mathbf{v}) - \mathbf{e}_i||^2$.]
   - Let M be an isometry with $M(\mathbf{0}) = \mathbf{0}$. Show that $\mathbf{v}_i = M(\mathbf{e}_i)$ is a unit vector and $\mathbf{v}_i \perp \mathbf{v}_j$. [Hint: another expansion.] Using part 1, argue that M must be the linear transformation determined by $M(\mathbf{e}_i) = \mathbf{v_i}$.
   - Show that M is an isometry $\Leftrightarrow$ M can be written as $Q\mathbf{x} + \mathbf{a}$, where $\mathbf{a} = M(\mathbf{0})$ and Q is an orthogonal matrix, i.e. $Q^tQ = 1$. Note this implies $\det Q = \pm 1$; we say M is direct if $\det Q = 1$ and opposite if $\det Q = -1$.
   - Show that an isometry of the plane can be expressed as either $M(x) = \rho_\theta \mathbf{x} + T_{\mathbf{a}}$ or $\rho_\theta \circ r(\mathbf{x}) + T_{\mathbf{a}}$, where $T_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ is translation by $\mathbf{a}$, ρ_θ is rotation counterclockwise by θ, and r is a reflection across the x-axis. [Hint: What are the matrices of these transformations? Compare to the orthogonal matrices of dimension two.]

   Note that the composition $\rho_\theta \circ r$ can also be described a reflection across a line which intersects the x-axis at an angle θ.